

# SPACEABILITY AND OPTIMAL ESTIMATES FOR SUMMING MULTILINEAR OPERATORS

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**ABSTRACT.** We show that given a positive integer  $m$ , a real number  $p \in [2, \infty)$  and  $1 \leq s < p^*$  the set of non-multiple  $(r, s)$ -summing  $m$ -linear forms on  $\ell_p \times \cdots \times \ell_p$  is spaceable whenever  $r < \frac{2ms}{s+2m-ms}$ . This result is optimal since for  $r \geq \frac{2ms}{s+2m-ms}$  all  $m$ -linear forms on  $\ell_p \times \cdots \times \ell_p$  are multiple  $(r, s)$ -summing. Among other results, we improve some results from [24] and generalize a result related to cotype (from 2010) due to Botelho, Michels and the second named author. We also prove some new coincidence results for the class of absolutely summing multilinear operators.

## 1. INTRODUCTION

The family of inequalities known as Bohnenblust–Hille, Littlewood’s 4/3 and Hardy–Littlewood (see [7, 18, 19]) dates back to the 30s and, after a long period of dormancy, have been rediscovered in the recent years with interesting applications in different fields. In the modern terminology, these inequalities can be seen as coincidence results in the theory of multiple summing operators. The main goal of this note is to investigate in details how the new advances in the study of the aforementioned inequalities can be explored in the context of multiple summing operators. We are also interested in estimating the size of the set of non multiple summing (and non absolutely summing) multilinear operators, and for this task we use the notion of spaceability.

Let  $E, E_1, \dots, E_m$  and  $F$  denote Banach spaces over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let  $B_{E^*}$  denote the closed unit ball of the topological dual of  $E$ . If  $1 \leq q \leq \infty$ , the symbol  $q^*$  represents the conjugate of  $q$ . For  $s > 0$ , by  $\ell_s(E)$  we mean the space of absolutely  $s$ -summable sequences in  $E$ ; we represent by  $\ell_s^w(E)$  the linear space of the sequences  $(x_j)_{j=1}^\infty$  in  $E$  such that  $(\varphi(x_j))_{j=1}^\infty \in \ell_s$  for every continuous linear functional  $\varphi : E \rightarrow \mathbb{K}$ . The function

$$\left\| (x_j)_{j=1}^\infty \right\|_{w,s} = \sup_{\varphi \in B_{E^*}} \left\| (\varphi(x_j))_{j=1}^\infty \right\|_s$$

defines on  $\ell_s^w(E)$  a  $s$ -norm if  $0 < s < 1$  and a norm if  $s \geq 1$ . The space of all continuous  $m$ -linear operators  $T : E_1 \times \cdots \times E_m \rightarrow F$ , with the sup norm, is denoted by  $\mathcal{L}(E_1, \dots, E_m; F)$ .

The following concept is a natural extension of the notion of absolutely summing linear operators (see [15, 21, 25]):

**Definition 1.1.** Let  $1 \leq s \leq r < \infty$ . A multilinear operator  $T \in \mathcal{L}(E_1, \dots, E_m; F)$  is multiple  $(r; s)$ -summing if there exists a  $C > 0$  such that

$$\left( \sum_{j_1, \dots, j_m=1}^\infty \left\| T(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)}) \right\|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^m \left\| (x_{j_k}^{(k)})_{j_k=1}^\infty \right\|_{w,s}$$

for all  $(x_{j_k}^{(k)})_{j_k=1}^\infty \in \ell_s^w(E_k)$ ,  $k \in \{1, \dots, m\}$ . We represent the class of all multiple  $(r; s)$ -summing operators from  $E_1, \dots, E_m$  to  $F$  by  $\Pi_{\text{mult}(r,s)}(E_1, \dots, E_m; F)$  and  $\pi_{(r,s)}(T)$  denotes the infimum over all  $C$  as above.

The classical Bohnenblust–Hille inequality [7], in the modern terminology, can be stated in terms of multiple summing operators (see [9, 26]):

**Theorem 1.2** (Bohnenblust–Hille). *Every continuous  $m$ -linear operator  $T \in \mathcal{L}(E_1, \dots, E_m; \mathbb{K})$  is multiple  $\left(\frac{2m}{m+1}; 1\right)$ -summing for all Banach spaces  $E_1, \dots, E_m$  and  $\frac{2m}{m+1}$  is optimal.*

The paper is organized as follows. In Section 2, we prove one of our main results which, in particular, shows that if  $1 < s < p^*$ , the set  $\mathcal{L}({}^m\ell_p; \mathbb{K}) \setminus \Pi_{\text{mult}(\frac{2m}{m+1}, s)}({}^m\ell_p; \mathbb{K})$  contains, except for the null vector, a closed infinite dimensional Banach space. In Section 3 we show some consequences of the result of the previous section. For

instance, as a particular case of our main result we observe a new optimality component of the Bohnenblust–Hille inequality: the term 1 from the pair  $\left(\frac{2m}{m+1}; 1\right)$  is also optimal. In Section 4 we investigate the optimality of coincidence results for multiple summing operators in  $c_0$  and in Section 5 we investigate the optimality of coincidence results in the framework of absolutely summing multilinear operators. More precisely we obtain estimates for  $m$ -linear operators and  $m$ -homogeneous polynomials with  $m$  odd and as consequence we show that the Defant–Voigt theorem is optimal.

## 2. SPACEABILITY AND MULTIPLE SUMMABILITY

For a given Banach space  $E$ , a subset  $A \subset X$  is spaceable if  $A \cup \{0\}$  contains a closed infinite dimensional subspace of  $E$  (see [5, 14, 6]). It will be convenient to adopt that  $\frac{c}{\infty} = 0$  and  $\frac{c}{0} = \infty$  for any  $c > 0$ .

**Theorem 2.1.** *Let  $m \geq 1$ ,  $p \in [2, \infty)$ . If  $1 \leq s < p^*$  and*

$$r < \frac{2ms}{s + 2m - ms}$$

*then*

$$\mathcal{L}(^m \ell_p; \mathbb{K}) \setminus \prod_{\text{mult}(r,s)}(^m \ell_p; \mathbb{K})$$

*is spaceable.*

*Proof.* We consider the case of complex scalars. The case of real scalars is obtained from the complex case via a standard complexification argument (see [9]). An extended version of the Kahane–Salem–Zygmund inequality (see [3, Lemma 6.1]) asserts that if  $m, n \geq 1$  and  $p \in [2, \infty]$ , then there exists a  $m$ -linear map  $A_n : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{K}$  of the form

$$(1) \quad A_n(z^{(1)}, \dots, z^{(m)}) = \sum_{j_1, \dots, j_m=1}^n \pm z_{j_1}^{(1)} \cdots z_{j_m}^{(m)}$$

such that

$$\|A_n\| \leq C_m n^{\frac{m+1}{2} - \frac{m}{p}}$$

for some constant  $C_m > 0$ .

Let

$$\beta := \frac{p + s - ps}{ps}.$$

Observe that  $s < p^*$  implies  $\beta > 0$ . We have

$$\left( \sum_{j_1, \dots, j_m=1}^n \left| A_n \left( \frac{e_{j_1}}{j_1^\beta}, \dots, \frac{e_{j_m}}{j_m^\beta} \right) \right|^r \right)^{\frac{1}{r}} \leq \pi_{(r,s)}(A_n) \left\| \left( \frac{e_j}{j^\beta} \right)_{j=1}^n \right\|_{w,s}^m$$

i.e.,

$$\left( \sum_{j_1, \dots, j_m=1}^n \left| \frac{1}{j_1^\beta \cdots j_m^\beta} \right|^r \right)^{\frac{1}{r}} \leq \pi_{(r,s)}(A_n) \left\| \left( \frac{e_j}{j^\beta} \right)_{j=1}^n \right\|_{w,s}^m.$$

But, for  $n \geq 2$ , since  $\frac{1}{\beta s} + \frac{1}{\frac{p^*}{s}} = 1$ , we obtain

$$\begin{aligned} \left\| \left( \frac{e_j}{j^\beta} \right)_{j=1}^n \right\|_{w,s} &= \sup_{\varphi \in B_{\ell_{p^*}}} \left( \sum_{j=1}^n \left| \varphi \left( \frac{e_j}{j^\beta} \right) \right|^s \right)^{\frac{1}{s}} \\ &= \sup_{\varphi \in B_{\ell_{p^*}}} \left( \sum_{j=1}^n |\varphi_j|^s \frac{1}{j^{\beta s}} \right)^{\frac{1}{s}} \\ &\leq \left( \left( \sum_{j=1}^n |\varphi_j|^{p^*} \right)^{\frac{s}{p^*}} \left( \sum_{j=1}^n \frac{1}{j} \right)^{\beta s} \right)^{\frac{1}{s}} \\ &< (1 + \log n)^\beta. \end{aligned}$$

Hence

$$\left( \sum_{j=1}^n \frac{1}{j^{r\beta}} \right)^{\frac{m}{r}} \leq \pi_{(r,s)}(A_n) (1 + \log n)^{m\beta}$$

and, since  $r\beta \neq 1$  (because, otherwise,  $p < 2$ ),

$$(n^{1-r\beta})^{\frac{m}{r}} \leq \pi_{(r,s)}(A_n) (1 + \log n)^{m\beta}.$$

Since  $\|A_n\| \leq C_m n^{\frac{m+1}{2} - \frac{m}{p}}$  we have

$$\frac{\pi_{(r,s)}(A_n)}{\|A_n\|} \geq \frac{n^{\frac{m}{r} - (\frac{p+s-ps}{ps})m}}{(1 + \log n)^{m\beta} C_m n^{\frac{m+1}{2} - \frac{m}{p}}}.$$

By making  $n \rightarrow \infty$  and using that  $r < \frac{2ms}{s+2m-ms}$  we get

$$\lim_{n \rightarrow \infty} \frac{\pi_{(r,s)}(A_n)}{\|A_n\|} = \infty$$

and from the Open Mapping Theorem we conclude that  $\prod_{\text{mult}(r,s)}({}^m\ell_p; \mathbb{K})$  is not closed in  $\mathcal{L}({}^m\ell_p; \mathbb{K})$ . Now the result follows from [17, Theorem 5.6 and its reformulation] (see also [20]).  $\square$

### 3. SOME CONSEQUENCES

The following result is a simple consequence of Theorem 2.1.

**Corollary 3.1.** *Let  $m \geq 1$  and  $r \in \left[\frac{2m}{m+1}, 2\right]$ . Then*

$$\sup \left\{ s : \mathcal{L}({}^m\ell_p; \mathbb{K}) = \prod_{\text{mult}(r,s)}({}^m\ell_p; \mathbb{K}) \right\} \leq \frac{2mr}{mr + 2m - r}$$

for all  $2 \leq p < \frac{2mr}{r+mr-2m}$ .

*Proof.* Since  $\frac{2m}{m+1} \leq r \leq 2 < 2m$ , it follows that  $1 \leq \frac{2mr}{mr+2m-r}$  and  $2 < \frac{2mr}{r+mr-2m}$ . Note that

$$s > \frac{2mr}{mr + 2m - r}$$

implies

$$r < \frac{2ms}{s + 2m - ms}.$$

Therefore, for  $2 \leq p < \frac{2mr}{r+mr-2m}$ , from Theorem 2.1 we know that

$$\mathcal{L}({}^m\ell_p; \mathbb{K}) \setminus \prod_{\text{mult}(r,s)}({}^m\ell_p; \mathbb{K})$$

is spaceable for all  $\frac{2mr}{mr+2m-r} < s < p^*$ . In particular, for  $2 \leq p < \frac{2mr}{r+mr-2m}$ ,

$$\sup \left\{ s : \mathcal{L}({}^m\ell_p; \mathbb{K}) = \prod_{\text{mult}(r,s)}({}^m\ell_p; \mathbb{K}) \right\} \leq \frac{2mr}{mr + 2m - r}.$$

$\square$

This corollary together with our main result ensures that, for  $r \in \left[\frac{2m}{m+1}, 2\right]$  and  $2 \leq p < \frac{2mr}{r+mr-2m}$ ,

$$\sup \left\{ s : \mathcal{L}({}^m\ell_p; \mathbb{K}) = \prod_{\text{mult}(r,s)}({}^m\ell_p; \mathbb{K}) \right\} = \frac{2mr}{mr + 2m - r}.$$

When  $p = 2$  the expression above recovers the optimality of [9, Theorem 5.14] in the case of  $m$ -linear operators on  $\ell_2 \times \cdots \times \ell_2$ .

In 2010 G. Botelho, C. Michels and D. Pellegrino [9] have shown that for  $m \geq 1$  and Banach spaces  $E_1, \dots, E_m$  of cotype 2,

$$\mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}(2; \frac{2m}{2m-1})}(E_1, \dots, E_m; \mathbb{K})$$

and for Banach spaces of cotype  $k > 2$ ,

$$\mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}(2; \frac{km}{km-1} - \epsilon)}(E_1, \dots, E_m; \mathbb{K})$$

for all sufficiently small  $\epsilon > 0$ .

We now remark that it is not necessary to make any assumptions on the Banach spaces  $E_1, \dots, E_m$  and  $\frac{2m}{2m-1}$  is optimal. Given  $k > 2$ , in [24, page 194] it is said that it is not known if  $s = \frac{km}{km-1}$  is attained or not in

$$\sup\{s : \mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}(2;s)}(E_1, \dots, E_m; \mathbb{K}) \text{ for all } E_j \text{ of cotype } k\} \geq \frac{km}{km-1}.$$

The optimality of  $\frac{2m}{2m-1}$  ensures that  $s = \frac{km}{km-1}$  is not attained and thus improves the estimate of [24, Corollary 3.1], which can be improved to

$$\sup\{s : \mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}(2;s)}(E_1, \dots, E_m; \mathbb{K}) \text{ for all } E_j \text{ of cotype } k\} \in \left[ \frac{2m}{2m-1}, \frac{2km}{2km+k-2m} \right]$$

if  $k > 2$  and  $m \geq k$  is a positive integer.

More precisely we prove the following more general result. We remark that the part (i) of the above theorem can be also derived from [4, 16], although it is not explicitly written in the aforementioned papers:

**Theorem 3.2.** *Let  $m \geq 1$  and let  $E_1, \dots, E_m$  be Banach spaces.*

(i) *If  $r \in \left[ \frac{2m}{m+1}, 2 \right]$ , then*

$$\mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}\left(r, \frac{2mr}{mr+2m-r}\right)}(E_1, \dots, E_m; \mathbb{K}).$$

*and the value  $\frac{2mr}{mr+2m-r}$  is optimal.*

(ii) *If  $r \in (2, \infty)$ , the optimal value of  $s$  such that*

$$\mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}(r,s)}(E_1, \dots, E_m; \mathbb{K})$$

*belongs to  $\left[ \frac{2m}{2m-1}, \frac{m}{m-1} \right)$ .*

*Proof.* For  $q \geq 1$ , let  $X_q = \ell_q$  and let us define  $X_\infty = c_0$ . Let

$$q := \frac{2mr}{r + mr - 2m}.$$

Since  $r \in \left[ \frac{2m}{m+1}, 2 \right]$  we have that  $q \in [2m, \infty]$ . Since

$$\frac{m}{q} \leq \frac{1}{2} \quad \text{and} \quad r = \frac{2m}{m+1 - \frac{2m}{q}},$$

from the multilinear Hardy–Littlewood inequality (see, for example, [3, 18, 28]) there is a constant  $C > 0$  such that

$$\left( \sum_{j_1, \dots, j_m=1}^{\infty} |A(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq C \|A\|$$

for all continuous  $m$ -linear mapping  $A : X_q \times \dots \times X_q \rightarrow \mathbb{K}$ . Let  $T \in \mathcal{L}(E_1, \dots, E_m; \mathbb{K})$  and  $\left( x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \in \ell_{q^*}^w(E_k)$ ,  $k = 1, \dots, m$ . Now we use a standard argument (see [4]) to lift the result from  $X_q$  to arbitrary Banach spaces. From [15, Proposition 2.2] we know that exist an continuous linear operator  $u_k : X_q \rightarrow E_k$  so that  $u_k \cdot e_{j_k} = x_{j_k}^{(k)}$  and

$$\|u_k\| = \left\| \left( x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \right\|_{w, q^*}$$

for all  $k = 1, \dots, m$ . Therefore,  $S : X_q \times \dots \times X_q \rightarrow \mathbb{K}$  defined by  $S(y_1, \dots, y_m) = T(u_1 \cdot y_1, \dots, u_m \cdot y_m)$  is  $m$ -linear continuous operator and

$$\|S\| \leq \|T\| \prod_{k=1}^m \|u_k\| = \prod_{k=1}^m \left\| \left( x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \right\|_{w, q^*}.$$

Hence

$$\left( \sum_{j_1, \dots, j_m=1}^{\infty} \left| T \left( x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)} \right) \right|^r \right)^{\frac{1}{r}} \leq C \|T\| \prod_{k=1}^m \left\| \left( x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \right\|_{w, q^*},$$

and, as  $q^* = \frac{2mr}{mr+2m-r}$ , the last inequality proves that, for all  $m \geq 1$  and  $r \in \left[ \frac{2m}{m+1}, 2 \right]$ ,

$$\mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}\left(r, \frac{2mr}{mr+2m-r}\right)}(E_1, \dots, E_m; \mathbb{K}).$$

Now let us prove the optimality. From what we have just proved, for  $r \in \left[\frac{2m}{m+1}, 2\right]$ ,

$$U_{m,r} = \sup \left\{ s : \mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{mult}(r;s)}(E_1, \dots, E_m; \mathbb{K}) \text{ for all Banach spaces } E_j \right\} \geq \frac{2mr}{mr + 2m - r}.$$

From Corollary 3.1 we have, for  $2 \leq p < \frac{2mr}{r+mr-2m}$ ,

$$\sup \left\{ s : \mathcal{L}({}^m\ell_p; \mathbb{K}) = \prod_{\text{mult}(r;s)}({}^m\ell_p; \mathbb{K}) \right\} \leq \frac{2mr}{mr + 2m - r}.$$

Therefore,

$$U_{m,r} \leq \sup \left\{ s : \mathcal{L}({}^m\ell_p; \mathbb{K}) = \prod_{\text{mult}(r;s)}({}^m\ell_p; \mathbb{K}) \right\} \leq \frac{2mr}{mr + 2m - r}.$$

and we conclude that  $U_{m,r} = \frac{2mr}{mr+2m-r}$ .

In the case  $r > 2$ , consider the  $m$ -linear operator

$$\begin{aligned} T &: \ell_m \times \dots \times \ell_m \rightarrow \mathbb{K} \\ (x^{(1)}, \dots, x^{(m)}) &\mapsto T(x^{(1)}, \dots, x^{(m)}) = \sum_{j=1}^{\infty} x_j^{(1)} \dots x_j^{(m)}. \end{aligned}$$

Observe that

$$\left| T(x^{(1)}, \dots, x^{(m)}) \right| \leq \sum_{j=1}^{\infty} |x_j^{(1)} \dots x_j^{(m)}| \leq \left( \sum_{j=1}^{\infty} |x_j^{(1)}|^m \right)^{\frac{1}{m}} \dots \left( \sum_{j=1}^{\infty} |x_j^{(m)}|^m \right)^{\frac{1}{m}}$$

and thus  $\|T\| \leq 1$ . Since  $(e_j) \in \ell_{\frac{m}{m-1}}^w(\ell_m)$  and  $T(e_j, \dots, e_j) = 1$  for all  $j$ , we conclude that  $T$  is not multiple  $(r; \frac{m}{m-1})$ -summing regardless of the choice of  $r$ . Therefore, keeping the notation of (i), we conclude that  $U_{m,r} \leq \frac{m}{m-1}$ . From the part (i) we have  $U_{m,2} = \frac{2m}{2m-1}$ . Thus, for  $r > 2$ ,  $U_{m,r} \geq \frac{2m}{2m-1}$  and the proof is done.  $\square$

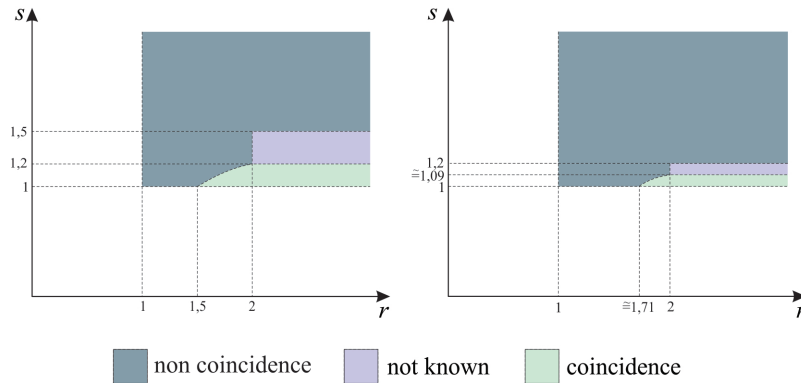


FIGURE 1. Coincidence zone for  $m = 3$  and  $m = 6$ , respectively

The table below details the results of coincidence and non coincidence in the “boundaries” of Figure 1 and we can clearly see that the only case that remains open is the case  $(r, s)$  with  $r > 2$  and  $\frac{2m}{2m-1} < s < \frac{m}{m-1}$ .

$r = 1$	$s \geq 1$	non coincidence
$1 \leq r < \frac{2m}{m+1}$	$s = 1$	non coincidence
$\frac{2m}{m+1} \leq r \leq 2$	$s = \frac{2mr}{mr+2m-r}$	coincidence
$r \geq \frac{2m}{m+1}$	$s = 1$	coincidence
$r = 2$	$\frac{2m}{2m-1} < s \leq \frac{m}{m-1}$	non coincidence
$r \geq 2$	$s = \frac{2m}{2m-1}$	coincidence
$r \geq 2$	$s = \frac{m}{m-1}$	non coincidence

#### 4. MULTIPLE $(r; s)$ -SUMMING FORMS IN $c_0$ AND $\ell_\infty$ SPACES

From standard localization procedures, coincidence results for  $c_0$  and  $\ell_\infty$  are the same; so we will restrict our attention to  $c_0$ . It is well known that  $\prod_{\text{mult}(r,s)}(^m c_0; \mathbb{K}) = \mathcal{L}(^m c_0; \mathbb{K})$  whenever  $r \geq s \geq 2$  (see [9]). When  $s = 1$ , as a consequence of the Bohnenblust–Hille inequality, we also know that the equality holds if and only if  $s \geq \frac{2m}{m+1}$ . The next result encompasses essentially all possible cases:

**Proposition 4.1.** *If  $s \in [1, \infty)$  then*

$$\inf \left\{ r : \prod_{\text{mult}(r,s)}(^m c_0; \mathbb{K}) = \mathcal{L}(^m c_0; \mathbb{K}) \right\} = \begin{cases} \frac{2m}{m+1} & \text{if } 1 \leq s \leq \frac{2m}{m+1} \\ s & \text{if } s \geq \frac{2m}{m+1}. \end{cases}$$

*Proof.* The case  $s \geq 2$  is immediate, since for  $r \geq s \geq 2$  we have coincidences and for  $r < s$  we obviously can not have coincidences.

The Bohnenblust–Hille inequality assures that when  $s = 1$  the best choice for  $r$  is  $\frac{2m}{m+1}$ . So, it is obvious that for  $1 \leq s \leq \frac{2m}{m+1}$  the best value for  $r$  is not smaller than  $\frac{2m}{m+1}$ . More precisely,

$$\prod_{\text{mult}(r,s)}(^m c_0; \mathbb{K}) \neq \mathcal{L}(^m c_0; \mathbb{K})$$

whenever  $(r, s) \in \left[1, \frac{2m}{m+1}\right] \times \left[1, \frac{2m}{m+1}\right]$ . For linear operators a deep result due to Maurey and Pisier (see [15]) alerts us that the notions of absolutely  $(r, 1)$ -summing operators and  $(r, s)$ -summing operators coincide when  $s < r$ . An adaptation of this result to multiple summing operators (see [27, Theorem 3.16] or [9, Lemma 5.2]) combined with the coincidence result for  $(r, s) = \left(\frac{2m}{m+1}, 1\right)$  tells us that we also have coincidences for  $\left(\frac{2m}{m+1}, s\right)$  for all  $1 < s < \frac{2m}{m+1}$ . The remaining case  $(r, s)$  with  $\frac{2m}{m+1} < s < 2$  follows from an interpolation procedure in the lines of [9].  $\square$

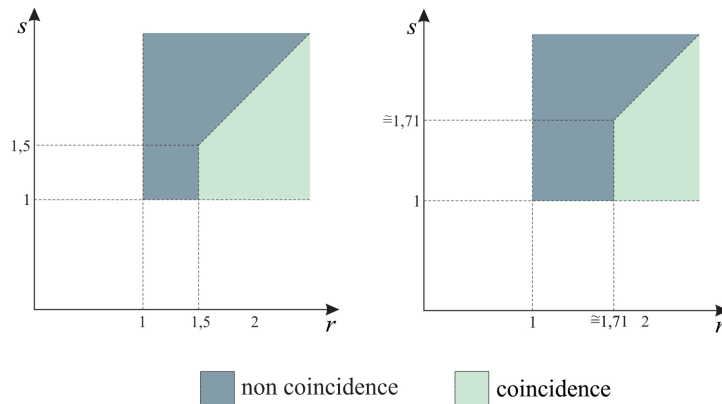


FIGURE 2. Coincidence zone for  $m = 3$  and  $m = 6$ , respectively

The table below details the results of coincidence and non coincidence in the “boundaries” of Figure 2.

$r = 1$	$s \geq 1$	non coincidence
$1 \leq r < \frac{2m}{m+1}$	$s = 1$	non coincidence
$r = \frac{2m}{m+1}$	$1 \leq s < \frac{2m}{m+1}$	coincidence
$r \geq \frac{2m}{m+1}$	$s = 1$	coincidence
$\frac{2m}{m+1} \leq r < 2$	$s = r$	not known
$r \geq 2$	$s = r$	coincidence

We can see that the only case that remains open is the case  $(r, s)$  with  $\frac{2m}{m+1} \leq r < 2$  and  $s = r$ .

### 5. ABSOLUTELY SUMMING POLYNOMIALS AND MULTILINEAR OPERATORS

For  $1 \leq s < \infty$  and  $r \geq \frac{s}{m}$  recall that a continuous  $m$ -linear operator  $A : E_1 \times \cdots \times E_m \rightarrow F$  is absolutely  $(r; s)$ -summing if there is a  $C > 0$  such that

$$\left( \sum_{j=1}^n \|A(x_j^{(1)}, \dots, x_j^{(m)})\|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^m \sup_{\varphi \in B_{E_k^*}} \left( \sum_{j=1}^n |\varphi(x_j^{(k)})|^s \right)^{\frac{1}{s}}$$

for all positive integers  $n$  and all  $(x_j^{(k)})_{j=1}^n \in E_k$ ,  $k = 1, \dots, m$ . We represent the class of all absolutely  $(r; s)$ -summing multiple operators from  $E_1, \dots, E_m$  to  $F$  by  $\Pi_{as(r; s)}(E_1, \dots, E_m; F)$ .

It is important to note that for  $(r; s)$  with  $s \geq \frac{m}{m-1}$ , there are Banach spaces  $E_1, \dots, E_m$  such that

$$\Pi_{as(r; s)}(E_1, \dots, E_m; F) \neq \mathcal{L}(E_1, \dots, E_m; F).$$

In fact, the same proof of the case (ii) of Theorem 3.2 shows that

$$\sup \left\{ s : \mathcal{L}({}^m \ell_m; \mathbb{K}) = \Pi_{as(r; s)}({}^m \ell_m; \mathbb{K}) \right\} \leq \frac{m}{m-1}.$$

Combining the Defant–Voigt Theorem (first stated and proved in [1, Theorem 3.10]; see also, e.g., [2, Theorem 3] (for complex scalars) or [11, Corollary 3.2]) and a canonical inclusion theorem (see [22]) we obtain that, for  $r, s \geq 1$  and  $s \leq \frac{mr}{mr+1-r}$ ,

$$\Pi_{as(r; s)}(E_1, \dots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \dots, E_m; \mathbb{K}).$$

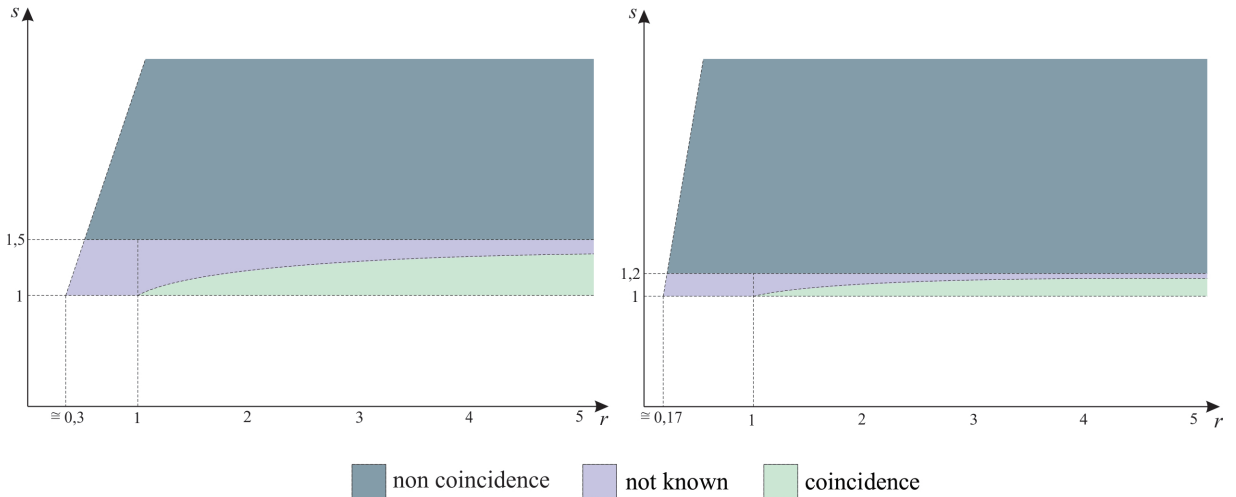


FIGURE 3. Coincidence zone for  $m = 3$  and  $m = 6$ , respectively

As a particular case of our results of this section we will improve the information contained in the above graphic.

Recall that, for  $1 \leq s < \infty$  and  $r \geq \frac{s}{m}$ , a continuous  $m$ -homogeneous polynomial  $P : E \rightarrow F$  is absolutely  $(r; s)$ -summing if there is a  $C > 0$  such that

$$\left( \sum_{j=1}^n \|P(x_j)\|^r \right)^{\frac{1}{r}} \leq C \sup_{\varphi \in B_{E^*}} \left( \sum_{j=1}^n |\varphi(x_j)|^s \right)^{\frac{m}{s}}$$

for all positive integers  $n$  and all  $x_j \in E$ . We will represent the class of all absolutely  $(r; s)$ -summing polynomials from  $E$  to  $F$  by  $\mathcal{P}_{\text{as}(r;s)}(^m E; F)$ .

It is obvious that to each coincidence result for absolutely summing multilinear operators corresponds a coincidence result for absolutely summing polynomials, but the validity of the converse seems to be an open problem. In this direction, by using polarization it is true that a coincidence result for absolutely summing polynomials implies in a correspondent coincidence result for absolutely summing symmetric multilinear operators.

It is well-known that every continuous scalar-valued  $m$ -homogeneous polynomial is absolutely  $(1; 1)$ -summing. More precisely

$$\mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(1;1)}(^m E; \mathbb{K})$$

regardless of the Banach space  $E$  (Defant–Voigt Theorem for polynomials). However, under certain cotype assumptions on  $E$ , even stronger results hold. More precisely, from [8] we know that if  $E$  has cotype  $k < m$ , then

$$\mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(\frac{k}{m}; 1)}(^m E; \mathbb{K}).$$

On the other hand, if  $m \geq 3$  it was recently shown (see [12]) that if  $\dim E = \infty$ , then

$$\mathcal{P}(^m E; \mathbb{K}) \neq \mathcal{P}_{\text{as}(\frac{1}{m}; 1)}(^m E; \mathbb{K}).$$

So, a natural problem is:

*For a given  $m \geq 1$ , what is the infimum of the  $r$  such that  $\mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K})$  for all infinite-dimensional Banach spaces  $E$ ?*

From the previous results we know that, for  $m \geq 3$ ,

$$(2) \quad \inf \{r : \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K})\} \in \left[ \frac{1}{m}, \min \left\{ \frac{\cot E}{m}, 1 \right\} \right].$$

and

$$(3) \quad V_m := \inf \{r : \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K}) \text{ for all infinite-dimensional Banach spaces } E\} \in \left[ \frac{1}{m}, 1 \right].$$

If  $m$  is even, it is essentially known (see [13, Theorem 3.1]) that

$$\inf \{r : \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K})\} \geq \frac{\cot E}{m + \cot E}.$$

However, the argument of the proof relies on some positiveness, which can be used just when  $m$  is even, and first working over  $\mathbb{K} = \mathbb{R}$ . The case of complex scalars is obtained by complexification. Therefore, if  $m$  is even, the estimate (2) can be improved to

$$\inf \{r : \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K})\} \in \left[ \frac{\cot E}{m + \cot E}, \min \left\{ \frac{\cot E}{m}, 1 \right\} \right].$$

and for (3) we have a definitive answer:

**Proposition 5.1.** *If  $m$  is even, then  $V_m = 1$ .*

*Proof.* For  $m$  even and  $r < 1$ , consider  $p$  such that  $\frac{mr}{1-r} < p$ . Therefore,  $\text{id}_{\ell_p}$  is not absolutely  $(\frac{mr}{1-r}; 1)$ -summing and so, by [13, Theorem 3.1],

$$\mathcal{P}_{\text{as}(r;1)}(^m \ell_p; \mathbb{K}) \neq \mathcal{P}(^m \ell_p; \mathbb{K})$$

for all  $r < 1$ . Thus

$$V_m \geq \inf \{r : \mathcal{P}_{\text{as}(r;1)}(^m \ell_p; \mathbb{K}) = \mathcal{P}(^m \ell_p; \mathbb{K})\} = 1$$

and the proof is done.  $\square$



However, there is no estimate for the case  $m$  odd and arbitrary Banach space  $E$ . In this direction, for the case of arbitrary  $m$ , up to now we just have (partial) answers when  $E$  has an unconditional Schauder basis (see [10, 23]). In this section we show that if  $m \geq 1$  is odd and  $\dim E = \infty$  we have estimates similar to the case  $m$  even. More precisely, we have for  $m$  odd,

$$\inf \{r \leq 1 : \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K})\} \geq \frac{\cot E}{m+1+\cot E}.$$

This result gives us new information to improve Figure 3 (see Figure 4).

The following lemma is known in the framework of multilinear operators. Our result is slightly different, although of the proof is standard; we sketch the proof for the sake of completeness. We stress that *a priori* this result is not a consequence of the multilinear corresponding result since, as we mentioned before, coincidence results for polynomials imply in coincidence results for *symmetric* multilinear operators.

**Lemma 5.2.** *If every continuous  $m$ -homogeneous polynomial from  $E$  to  $F$  is absolutely  $(r; s)$ -summing, then every continuous symmetric  $(m+1)$ -linear forms from  $E^{m+1}$  to  $F$  is absolutely  $(r; s, \dots, s, 1)$ -summing.*

*Proof.* From the Polarization Formula we conclude that every continuous symmetric  $m$ -linear form from  $E^m$  to  $F$  is absolutely  $(r; s, \dots, s)$ -summing. Suppose that  $A : E^{m+1} \rightarrow F$  is a symmetric continuous  $(m+1)$ -linear form and let  $(x_j^{(k)})_{j=1}^n \in E$ ,  $k = 1, \dots, m+1$ . Let  $\varphi_j \in B_{F^*}$  be such that

$$(4) \quad \left\| A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right\| = \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right)$$

for all  $j = 1, \dots, n$ . Let also  $b_1, \dots, b_n$  be non negative real numbers such that  $\sum_{j=1}^n b_j^{r^*} = 1$  and

$$\left( \sum_{j=1}^n \left\| A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right\|^r \right)^{\frac{1}{r}} = \sum_{j=1}^n b_j \left\| A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right\| \stackrel{(4)}{=} \sum_{j=1}^n b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right).$$

Let  $r_j$  be the Rademacher functions; we have

$$\begin{aligned} & \int_0^1 \sum_{j=1}^n r_j(t_{m+1}) b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m)}, \sum_{j_{m+1}=1}^n r_{j_{m+1}}(t_{m+1}) x_{j_{m+1}}^{(m+1)} \right) \right) dt_{m+1} \\ &= \int_0^1 \left( \sum_{j, j_{m+1}=1}^n b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m)}, x_{j_{m+1}}^{(m+1)} \right) \right) r_j(t_{m+1}) r_{j_{m+1}}(t_{m+1}) \right) dt_{m+1} \\ &= \sum_{j, j_{m+1}=1}^n \left( b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m)}, x_{j_{m+1}}^{(m+1)} \right) \right) \int_0^1 r_j(t_{m+1}) r_{j_{m+1}}(t_{m+1}) dt_{m+1} \right) \\ &= \sum_{j=1}^n b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m)}, x_j^{(m+1)} \right) \right) \end{aligned}$$

and thus

$$\begin{aligned}
& \left( \sum_{j=1}^n \left\| A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right\|^r \right)^{\frac{1}{r}} \\
&= \sum_{j=1}^n b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right) \\
&= \int_0^1 \sum_{j=1}^n r_j(t_{m+1}) b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m)}, \sum_{j_{m+1}=1}^n r_{j_{m+1}}(t_{m+1}) x_{j_{m+1}}^{(m+1)} \right) \right) dt_{m+1} \\
&\leq \int_0^1 \left| \sum_{j=1}^n r_j(t_{m+1}) b_j \varphi_j \left( A \left( x_j^{(1)}, \dots, x_j^{(m)}, \sum_{j_{m+1}=1}^n r_{j_{m+1}}(t_{m+1}) x_{j_{m+1}}^{(m+1)} \right) \right) \right| dt_{m+1} \\
&\leq \int_0^1 \left( \sum_{j=1}^n b_j \left\| A \left( x_j^{(1)}, \dots, x_j^{(m)}, \sum_{j_{m+1}=1}^n r_{j_{m+1}}(t_{m+1}) x_{j_{m+1}}^{(m+1)} \right) \right\|^r \right) dt_{m+1} \\
&\leq \sup_{|t_{m+1}| \leq 1} \sum_{j=1}^n b_j \left\| A \left( x_j^{(1)}, \dots, x_j^{(m)}, \sum_{j_{m+1}=1}^n r_{j_{m+1}}(t_{m+1}) x_{j_{m+1}}^{(m+1)} \right) \right\|^r \\
&\leq \sup_{|t_{m+1}| \leq 1} \left( \sum_{j=1}^n \left\| A \left( x_j^{(1)}, \dots, x_j^{(m)}, \sum_{j_{m+1}=1}^n r_{j_{m+1}}(t_{m+1}) x_{j_{m+1}}^{(m+1)} \right) \right\|^r \right)^{\frac{1}{r}}.
\end{aligned}$$

Denoting

$$z_{m+1} = \sum_{j=1}^n r_j(t_{m+1}) x_j^{(m+1)},$$

note that by hypothesis the  $m$ -homogeneous polynomial

$$P : E \rightarrow F$$

$$x \mapsto P(x) = A(x, \dots, x, z_{m+1})$$

is absolutely  $(r; s)$ -summing and hence its associated symmetric multilinear operator is absolutely  $(r; s)$ -summing. Thus

$$\begin{aligned}
\left( \sum_{j=1}^n \left\| A \left( x_j^{(1)}, \dots, x_j^{(m+1)} \right) \right\|^r \right)^{\frac{1}{r}} &\leq \sup_{|t_{m+1}| \leq 1} \left( \sum_{j=1}^n \left\| A \left( x_j^{(1)}, \dots, x_j^{(m)}, \sum_{j_{m+1}=1}^n r_{j_{m+1}}(t_{m+1}) x_{j_{m+1}}^{(m+1)} \right) \right\|^r \right)^{\frac{1}{r}} \\
&\leq \sup_{|t_{m+1}| \leq 1} \left( \sum_{j=1}^n \left\| A \left( x_j^{(1)}, \dots, x_j^{(m)}, z_{m+1} \right) \right\|^r \right)^{\frac{1}{r}} \\
&\leq \sup_{|t_{m+1}| \leq 1} C \|A(\cdot, \dots, \cdot, z_{m+1})\| \left\| \left( x_j^{(1)} \right)_{j=1}^n \right\|_{w,s} \cdots \left\| \left( x_j^{(m)} \right)_{j=1}^n \right\|_{w,s} \\
&\leq \sup_{|t_{m+1}| \leq 1} C \|A\| \|z_{m+1}\| \left\| \left( x_j^{(1)} \right)_{j=1}^n \right\|_{w,s} \cdots \left\| \left( x_j^{(m)} \right)_{j=1}^n \right\|_{w,s} \\
&\leq C \|A\| \left\| \left( x_j^{(1)} \right)_{j=1}^n \right\|_{w,s} \cdots \left\| \left( x_j^{(m)} \right)_{j=1}^n \right\|_{w,s} \left\| \left( x_j^{(m+1)} \right)_{j=1}^n \right\|_{w,1}.
\end{aligned}$$

□

**Theorem 5.3.** *If  $r < 1$ ,  $\dim E = \infty$ ,  $\mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K})$  and  $m$  is odd, then  $\cot E < \infty$  and*

$$r \geq \frac{\cot E}{m+1 + \cot E}$$

*Proof.* From the previous result we have

$$\mathcal{P}(^{m+1} E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^{m+1} E; \mathbb{K}).$$

Since  $m+1$  is even, from [13, Theorem 3.1] we conclude that  $\text{id}_E$  is absolutely  $\left(\frac{(m+1)r}{1-r}; 1\right)$ -summing. So,  $\cot E < \infty$  and

$$r \geq \frac{\cot E}{m+1 + \cot E}.$$

□

We remark that if  $r \geq \frac{\cot E}{m}$  it is well known that the coincidence holds. So, for  $m$  odd,

$$\inf \left\{ r : \mathcal{P}(^m E; \mathbb{K}) = \mathcal{P}_{\text{as}(r;1)}(^m E; \mathbb{K}) \right\} \in \left[ \frac{\cot E}{m+1 + \cot E}, \min \left\{ 1, \frac{\cot E}{m} \right\} \right].$$

**Corollary 5.4.**  $V_m = 1$  for all positive integers  $m$ .

*Proof.* We just need prove the case  $m$  odd. For  $m$  odd and  $r < 1$ , consider  $p$  such that  $\frac{mr+r}{1-r} < p$ . Thus, it follows from Theorem 5.3 that

$$\mathcal{P}_{\text{as}(r;1)}(^m \ell_p; \mathbb{K}) \neq \mathcal{P}(^m \ell_p; \mathbb{K})$$

for all  $r < 1$ . Therefore

$$V_m \geq \inf \{ r : \mathcal{P}_{\text{as}(r;1)}(^m \ell_p; \mathbb{K}) = \mathcal{P}(^m \ell_p; \mathbb{K}) \} = 1.$$

□

As we already said, each coincidence result for absolutely summing multilinear operators corresponds a coincidence result for absolutely summing polynomials. Thus, the above corollary ensures that the zone defined by  $r < 1$  and  $s \geq 1$  in the Figure 3 is a non coincidence zone (see Figure 4). In particular, we note that the following result is consequence of the our previous results:

**Theorem 5.5.** *The Defant–Voigt Theorem is optimal. More precisely if  $m \geq 1$  is a positive integers,*

$$\inf \left\{ r : \mathcal{L}(E_1, \dots, E_m; \mathbb{K}) = \prod_{\text{as}(r;1)}(E_1, \dots, E_m; \mathbb{K}) \text{ for all infinite-dimensional Banach spaces } E_j \right\} = 1.$$

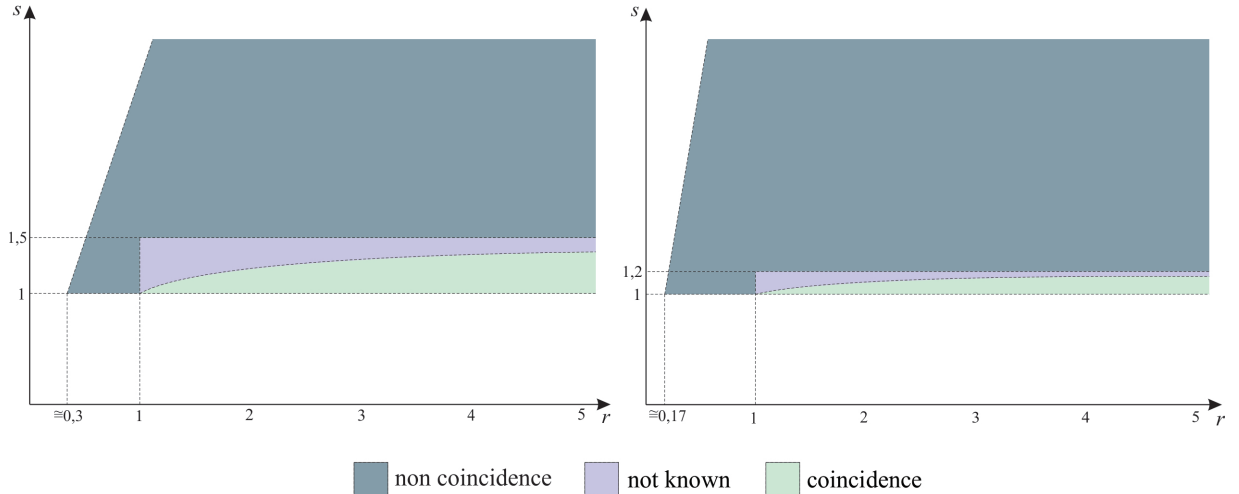


FIGURE 4. Coincidence zone for  $m = 3$  and  $m = 6$ , respectively

The table below details the results of coincidence and non coincidence in the “boundaries” of Figure 4.

$\frac{1}{m} \leq r < 1$	$s = 1$	non coincidence
$r \geq \frac{1}{m}$	$s = mr$	non coincidence
$r = 1$	$1 < s \leq \frac{m}{m-1}$	non coincidence
$r \geq 1$	$s = 1$	coincidence
$r \geq 1$	$s = \frac{mr}{mr+1-r}$	coincidence
$r \geq 1$	$s = \frac{m}{m-1}$	non coincidence

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